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# Fermions Interacting with Spherically Symmetric Monopoles: Beyond the Step Function Approximation

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ABSTRACT

We analyse the interactions of massless fermions in arbitrary representations of the gauge group, with general spherically symmetric monopoles of arbitrary strength. We obtain the conditions for finiteness of the solutions to the Dirac equation at the origin, evolve it through arbitrary radial functions and obtain the boundary conditions at the monopole core radius, thus improving upon the step function approximation previously used. We show that our results differ from the results using the step function approximation only when the boundary conditions are not diagonalisable on the physical basis, and hence processes involving such cases may be used to probe the structure of the monopole core.

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## I. INTRODUCTION

Ever since the theoretical possibility of monopole catalysis of baryon decay was discovered,[1,2] there has been a concerted effort to understand the ramifications of the monopole-fermion system.[3] The original work on the lowest dimensional representation of massless fermions interacting with the lowest strength monopole has been extended to higher dimensional representations of massless fermions[4,5] and to massive fermions.[6] Higher strength monopoles have also been discussed.[7] The next extension is to study fermions interacting with arbitrary spherically symmetric monopoles.[8] Schellekens[9] has made a systematic analysis of massless fermions in arbitrary representations of the gauge group interacting with spherically symmetric monopoles of arbitrary strength.

In this paper, we shall extend the work of Schellekens by relaxing the step function approximation that he used to obtain the boundary conditions. Since the boundary conditions depend crucially on the monopole core dynamics, it is not a priori obvious, that the step function approximation is justified, even in the limit where the core radius is vanishingly small. We find that by including an arbitrary radial function inside the monopole core, we introduce an arbitrariness in the boundary conditions, but they still remain unitary. Unitary boundary conditions are needed for the hermiticity of the truncated Hamiltonian,--i.e. the Hamiltonian obtained from the full Dirac Hamiltonian by replacing the monopole core dynamics by the boundary conditions. Since there exists a well-defined prescription for obtaining the boundary conditions at $r = r_0$ (where $r_0$ is the core radius) from the singularity structure at $r = 0$, we follow that prescription to show that the boundary conditions do imply a hermitean truncated Hamiltonian and consequently the boundary conditions are unitary.

Schellekens has shown that in the step function approximation, we may get the boundary conditions in terms of just the physically relevant fields--i.e. the fields that interact non-trivially with the monopole core. We show that even with arbitrary radial functions inside the core, the decoupling of the irrelevant fields occurs, and we get the right number of boundary conditions. In fact, in most physical cases, the boundary conditions in terms of the physical fields are one dimensional, so that the arbitrariness of the unitary matrix reduces to an arbitrary phase, and our results in terms of physical cross-sections are identical to those obtained in the step function approximation. Only in the somewhat unusual case, where the boundary conditions are not reducible to the one dimensional form, our results are observably different--i.e. physical cross-sections are numerically different--from the step function approximation. Such cases may be used to probe the monopole core.

The paper is organised in the following way. In Section II, we solve the Dirac equation at the origin in the presence of an arbitrary spherically symmetric monopole, closely following Schelleken's formalism. The purpose of this section is just to fix the notation and make the paper self-contained. In Section III, we evolve the conditions needed for finiteness of the solutions at the origin to $r = r_0$, and prove that they are unitary. In Section IV, we describe the decoupling of the irrelevant fields and obtain the effective boundary conditons.

Finally in Section V, we describe how the arbitrary unitary matrices are reduced to unobservable phases in the Green's functions, and offer our conclusions.

## II. BOUNDARY CONDITIONS ON THE FERMION FIELDS AT THE ORIGIN

In this section, we shall briefly review the formalism of spherically symmetric monopoles in order to fix our notation. Then we shall obtain the general form of the Dirac equation for fermions in all partial waves, in any representation of the gauge group, both inside and outside the monopole core. Then, we solve the equations of motion at the origin and obtain the condition on the fermion fields for finiteness of the solution.

### A) Spherically Symmetric Monopoles

Let us consider a group G broken down to a subgroup H, by the vacuum expectation value of a Higgs field $\phi$. The vector potential of the point monopole, in the string gauge, is given by

$$\vec{A} = Q\,\vec{A}_D \tag{2.1}$$

where the Dirac monopole field $\vec{A}_D$ is

$$\vec{A}_D = \frac{(1 - \cos\theta)\ \hat{\phi}}{r\sin\theta} \tag{2.2}$$

The matrix Q is a constant generator of H.

Now consider an SU(2) subgroup of G with generators $\vec{T}$. Goldhaber and Wilkinson[8] proved that a monopole field can be gauge transformed to a gauge where 2.1 becomes spherically symmetric under $\vec{L} + \vec{T}$ iff

$$Q = I_3 - T_3 \tag{2.3}$$

where $\vec{I}$ are the generators of an SU(2) subgroup of H and furthermore

$$[Q,\vec{I}] = 0 \tag{2.4}$$

In the gauge-transformed basis,

$$\vec{A} = \frac{(\vec{I}(\hat{r}) - \vec{T})\ x\hat{r}}{r} \tag{2.5}$$

where $\vec{I}(\hat{r})$ is defined from $\vec{I}$ by a suitable gauge transformation.

They also showed that for a finite energy non-singular solution, the most general spherically symmetric ansatz for $\vec{A}$ is

$$\vec{A} = \frac{(\vec{P}(r,\hat{r}) - \vec{T})\ x\hat{r}}{r} \tag{2.6}$$

where $\vec{P}(r,\hat{r})$ is a vector under $\vec{L} + \vec{T}$ and $\vec{P}(r,\hat{r}) = \vec{I}(\hat{r})$ for $r > r_0$.

Since this formalism has been described in detail in the original paper[8] and also by Schellekens,[9] we shall not go into further details here.

B) Form of the Dirac Equation

We start with the two-dimensional form of the Dirac operator, for the left-handed Weyl fields and we work in the $A_0 = 0$ gauge

$$-i\partial_0 = i\frac{D_\Omega}{r} + i\,\vec{\sigma}\cdot\hat{r}\,(\partial r + \frac{1}{r}) \tag{2.7}$$

where

$$D_\Omega = r\sigma_i\,(\delta_{ij} - \hat{r}_i\hat{r}_j)\partial_j - i\sigma_i\,\varepsilon_{iaj}\,(P_a(r,\hat{r}) - T_a)\,\hat{r}_j - \hat{r}\cdot\vec{\sigma} \tag{2.8}$$

with

$$P_a(r,\hat{r}) = T_a,\ r = 0 \tag{2.9}$$

$$= P_a(r,\hat{r}),\ 0 < r < r_0 \tag{2.10}$$

$$= I_a(\hat{r}),\ r > r_0 \tag{2.11}$$

$D_\Omega$ can be expanded in terms of the vector

$$\vec{M} = \vec{L} + \vec{S} + \vec{T} - \vec{P}(r,\hat{r}) \tag{2.12}$$

as

$$D_\Omega = -\vec{\sigma}\cdot\hat{r}\ (\vec{\sigma}\cdot\vec{M} - \vec{\sigma}\cdot\hat{r}\ \vec{M}\cdot\hat{r}) \tag{2.13}$$

The fermion fields can be expanded in eigen-states of $J^2$, $J_3$, $S^2$, $T^2$, $(I(\hat{r}))^2$, $\hat{r}\cdot\vec{I}(\hat{r})$ and $\hat{r}\cdot\vec{T}$. We replace $\hat{r}\cdot\vec{I}(\hat{r})$ by $\hat{Q} = \hat{r}\cdot\vec{I}(\hat{r}) - \hat{r}\cdot\vec{T}$ and label our states in this basis for given $\vec{T}$, $\vec{I}$ and $\vec{S} = 1/2$ by $|J^2,\ J_3,\ \hat{r}\cdot\vec{S},\ \hat{r}\cdot\vec{T},\ \hat{Q}>$. In the $\hat{r}\cdot\vec{T}$, $\hat{r}\cdot\vec{S}$ space, we order the fields as follows

$$\{\psi_{D_1,-1/2},\ \psi_{D_2,1/2}\} \tag{2.14}$$

where $\pm 1/2$ are the $\hat{r}\cdot\vec{S}$ eigenvalues and

$$D_1 = D_2 = +t,\ t-1\ .\ .\ .\ -t,\ \text{for } j \geq t + 1/2 \tag{2.15}$$

$$D_1 = +j + 1/2,\ (j-1) + 1/2\ .\ .\ .\ -j + 1/2$$

$$D_2 = -j - 1/2,\ (-j+1) - 1/2\ .\ .\ .\ +j - 1/2, \quad \text{for } j \leq t - 1/2 \tag{2.16}$$

are the $\hat{r}\cdot\vec{T}$ eigen-values. In this space, the operator $\vec{\sigma}\cdot\vec{M} - \vec{\sigma}\cdot\hat{r}\ \vec{M}\cdot\hat{r}$ can be written as

$$\vec{\sigma}\cdot\vec{M} - \vec{\sigma}\cdot\hat{r}\ \vec{M}\cdot\hat{r} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix} \tag{2.17}$$

where $A$ and $A^\dagger$ are matrices in the space of $D_1$ and $D_2$ respectively. Further, the phases of the fields can always be chosen so that at the origin,

$$A = A_0 = (\sqrt{J(J+1) - (\hat{r}\cdot\vec{T})^2 + 1/4}\ \mathbb{1} - \hat{I}^-)\ \hat{S}^+ \qquad (2.18)$$

and outside the monopole core

$$A = A_+ = (\sqrt{J(J+1) - (\hat{r}\cdot\vec{T})^2 + 1/4}\ \mathbb{1} - \hat{T}^-)\ \hat{S}^+ \qquad (2.19)$$

Here $\hat{T}^\pm$, $\hat{I}^\pm$ and $\hat{S}^\pm$ are the raising and lowering operators for $\hat{r}\cdot\vec{T}$, $\hat{r}\cdot\vec{I}$ and $\hat{r}\cdot\vec{S}$ respectively with standard angular momentum phase conventions (i.e., matrix elements of $\hat{T}^\pm$, $\hat{I}^\pm$ and $\hat{S}^\pm$ are real and positive.) Inside the monopole core, the operator structure depends on the unknown radial function $\vec{P}(r,\hat{r})$ and hence $A_-$ is a function of r.

C) Solution of the Dirac Equation at the Origin

The Dirac equation is given by equation 2.7 with the appropriate form of $D_\Omega$. We make explicit a 1/r dependence of the solution, so that $\partial r + 1/r$ can be replaced by $\partial r$ and work in the approximation where $H\chi = E\chi \approx 0$, so that equation 2.7 reduces to

$$\partial_r \begin{pmatrix} \psi_{D_1,-1/2} \\ \psi_{D_2,1/2} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} 0 & A_0 \\ A_0^\dagger & 0 \end{pmatrix} \begin{pmatrix} \psi_{D_1,-1/2} \\ \psi_{D_2,1/2} \end{pmatrix} \qquad (2.20)$$

Following Ref. 9, we decompose $A_0$ into a product of hermitean and unitary matrices

$$A_0 = H_0 U_0 \qquad (2.21)$$

$H_0$ may be diagonalised by a unitary transformation matrix $S_0$, and furthermore, the decomposition 2.21 can always be chosen so that the diagonal matrix $D_0$ has positive eigen-values.

Hence, we have

$$\partial_r \begin{pmatrix} S_0 & \psi_{D_1,-1/2} \\ S_0U_0 & \psi_{D_2,1/2} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} 0 & S_0A_0U_0^\dagger S_0^\dagger \\ S_0U_0A_0^\dagger S_0^\dagger & 0 \end{pmatrix} \begin{pmatrix} S_0 & \psi_{D_1,-1/2} \\ S_0U_0 & \psi_{D_2,1/2} \end{pmatrix}$$

$$= \frac{1}{r} \begin{pmatrix} 0 & D_0 \\ D_0 & 0 \end{pmatrix} \begin{pmatrix} S_0 & \psi_{D_1,-1/2} \\ S_0U_0 & \psi_{D_2,1/2} \end{pmatrix} \tag{2.22}$$

leading to

$$\partial_r \begin{pmatrix} S_0\psi_{D_1,-1/2} + S_0U_0\psi_{D_2,1/2} \\ S_0\psi_{D_1,-1/2} - S_0U_0\psi_{D_2,1/2} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} D_0 & 0 \\ 0 & -D_0 \end{pmatrix} \begin{pmatrix} S_0\psi_{D_1,-1/2} + S_0U_0\psi_{D_2,1/2} \\ S_0\psi_{D_1,-1/2} - S_0U_0\psi_{D_2,1/2} \end{pmatrix} \tag{2.23}$$

so that for finiteness of the solution at the origin, we have to set

$$\psi_{D_1,-1/2} = U_0\, \psi_{D_2,1/2} \tag{2.24}$$

where ,because of our phase conventions, $U_0$ is an orthogonal matrix. We may write this result in terms of a projection operator P acting on the column vector $\chi = (\psi_{D_1,-1/2}, \psi_{D_2,1/2})$ as

$$P\ (r = 0) = \begin{pmatrix} 1 & -U_0 \\ 0 & 0 \end{pmatrix} \tag{2.25}$$

## III. EVOLUTION OF THE BOUNDARY CONDITIONS

In this section, we shall evolve the conditions on the fermion fields for finiteness at the origin, through arbitrary radial functions to $r = r_0$. Since we cannot solve the equations of motion inside the core, we evolve the projection operator and show that it satisfies the condition for hermiticity of the truncated Hamiltonian, for any radial function inside the core.

Let us first find the necessary condition for the truncated Hamiltonian to be hermitean. Since $iD_\Omega = \begin{pmatrix} 0 & -iA \\ iA^\dagger & 0 \end{pmatrix}$ is obviously hermitean, the condition is

$$\int d^3x \; \psi_1^\dagger \; (-i \; \vec{\sigma}\cdot\hat{r} \; \partial r) \; \psi_2 = \int d^3x \; (-i \; \vec{\sigma}\cdot\hat{r} \, \partial r \; \psi_1)^\dagger \psi_2 \tag{3.1}$$

By choosing the fermions in the $(\psi_{D_1,-1/2}, \psi_{D_2,1/2})$ basis, we may write

$$\vec{\sigma}\cdot\hat{r} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv \Gamma_1 \tag{3.2}$$

To get the condition in a more convenient form, we integrate the L.H.S. of equation 3.1 by parts to get

$$\int_{\text{boundary}} d^2x \; \psi_1^\dagger(-i\Gamma_1) \; \psi_2 - \int d^3x \; (i\partial r \; \psi_1^\dagger) \; \Gamma_1\psi_2 \tag{3.3}$$

which is equal to the R.H.S. if

$$\psi_1^\dagger \, \Gamma_1 \, \psi_2 = 0 \quad \text{at } r = r_0 \tag{3.4}$$

Since, at the boundary, $\psi_1 = (1 - P)\,\psi_1$ and $\psi_2 = (1 - P)\,\psi_2$, the condition can be written in terms of the projection operators as

$$(1 - P)^\dagger \, \Gamma_1 \, (1 - P) = 0 \quad \text{at } r = r_0 \tag{3.5}$$

$P(r = 0)$ obviously satisfies the hermiticity condition and we wish to show that $P$ evolved to $r = r_0$ also does.

The equations of motion inside the core are

$$\partial_r \chi = M(r)\,\chi \tag{3.6}$$

where

$$M(r) = \frac{1}{r}\begin{pmatrix} 0 & A_- \\ A_-^\dagger & 0 \end{pmatrix}.$$

Let us transform to the basis

$$\Phi = \left(S_0 \psi_{D_1,-1/2} - S_0 U_0 \psi_{D_2,1/2} \;,\; S_0 \psi_{D_1,-1/2} + S_0 U_0 \psi_{D_2,1/2}\right) \tag{3.7}$$

where the projection operator at the origin is

$$P_\Phi(r = 0) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} . \tag{3.8}$$

$M(r)$ is transformed as well and

$$M_\Phi(r) = \frac{1}{2r}\begin{pmatrix} -S_0U_0A_-^\dagger S_0^\dagger - S_0A_-U_0^\dagger S_0^\dagger & -S_0U_0A_-^\dagger S_0^\dagger + S_0A_-U_0^\dagger S_0^\dagger \\ S_0U_0A_-^\dagger S_0^\dagger - S_0A_-U_0^\dagger S_0^\dagger & S_0U_0A_-^\dagger S_0^\dagger + S_0A_-U_0^\dagger S_0^\dagger \end{pmatrix} \tag{3.9}$$

In this basis, the solution at $r = r_0$ is

$$\Phi\ (r_0) = P\ e^{\int_\varepsilon^{r_0} M_\Phi(r)\,dr}\ \Phi\ (r = \varepsilon)$$

$$= O_\varepsilon\ \ \Phi\ (r = \varepsilon) \tag{3.10}$$

where P denotes path ordering and the regulator $\varepsilon$ is needed to take care of the singularity of $M_\Phi(r)$ at the origin.

Now, in the limit $\varepsilon \to 0$, $\Phi(r_0)$ is finite provided that the solution at the origin satisfies the appropriate finiteness condition--i.e. if

$$P_\Phi(r = 0)\ \ \Phi\ (0) = 0 \tag{3.11}$$

$$\Longrightarrow \qquad \lim_{\varepsilon \to 0}\ \ P_\Phi\ (r = \varepsilon)\ \ \Phi\ (r = \varepsilon) = 0 \tag{3.12}$$

Hence, $\lim_{\varepsilon \to 0} O_\varepsilon\ (1 - P_\Phi\ (r = \varepsilon))$ is finite--i.e. if we write

$$O_\varepsilon = \begin{pmatrix} O_1 & O_2 \\ O_3 & O_4 \end{pmatrix} \tag{3.13}$$

$O_2$ and $O_4$ are finite when $\varepsilon \to 0$.

Let us now construct the projection operator at $r = r_0$. We write $\Phi = (\phi_1, \phi_2)$ and use

$$\Phi\ (r = r_0) = \lim_{\varepsilon \to 0}\ O_\varepsilon\ (1 - P_\phi(r = \varepsilon)\ )\ \Phi\ (r = \varepsilon) \qquad (3.14)$$

$$\Longrightarrow \quad \begin{pmatrix} \Phi_1(r=r_0) \\ \Phi_2(r=r_0) \end{pmatrix} = \lim_{\epsilon \to 0} \begin{pmatrix} O_1 & O_2 \\ O_3 & O_4 \end{pmatrix} \begin{pmatrix} O & O \\ O & -I \end{pmatrix} \begin{pmatrix} \Phi_1(r=\epsilon) \\ \Phi_2(r=\epsilon) \end{pmatrix} \qquad (3.14a)$$

to obtain

$$\phi_1(r_0) = O_2 O_4^{-1} \phi_2(r_0) \qquad (3.15)$$

which gives

$$P(r = r_0) \equiv P_1(r = r_0) = \begin{pmatrix} 1 & -O_2 O_4^{-1} \\ 0 & 0 \end{pmatrix} \qquad (3.16)$$

Since $P_1(r = r_0)$ depends only on $O_2$ and $O_4$, it is explicitly finite in the $\varepsilon \to 0$ limit. The hermiticity of the Hamiltonian is not easy to see with this form of the projection operator. Hence, we construct another projection operator using

$$\lim_{\varepsilon \to 0}\ P(r = \varepsilon)\ \ \Phi(r = \varepsilon) = 0 \qquad (3.17)$$

$$\Longrightarrow \quad \lim_{\varepsilon \to 0} O_\varepsilon \; P(r = \varepsilon) \, O_\varepsilon^{-1} \, O_\varepsilon \, \Phi\,(r = \varepsilon) = 0 \tag{3.18}$$

Now $\lim_{\varepsilon \to 0} O_\varepsilon \, \Phi(r = \varepsilon)$ is finite, but

$$P(r = r_0) \equiv P_2(r = r_0) = O_\varepsilon \; P(r = \varepsilon) \; O_\varepsilon^{-1} \tag{3.19}$$

need not be finite in the $\varepsilon \to 0$ limit. In fact, we can explicitly show that

$$P_2(r = r_0) = A \;\; P_1(r = r_0) \tag{3.20}$$

where

$$A = \begin{pmatrix} O_1 & O_2 \\ O_3 & O_4 \end{pmatrix} \begin{pmatrix} (O_1 - O_2 O_4^{-1} O_3)^{-1} & O \\ 0 & I \end{pmatrix} \tag{3.21}$$

so that $P_2(r = r_0)$ and $P_1(r = r_0)$ project onto the same space, but $P_2(r = r_0)$ depends on $O_1$ and $O_3$ as well, and is not explicitly finite in the $\varepsilon \to 0$ limit.

We shall use $P_2(r = r_0)$ to prove the hermiticity of the Hamiltonian. We look at

$$(1 - P_2)^\dagger \Gamma_1 (1 - P_2) = [O(1 - P(r = \varepsilon))O^{-1}]^\dagger \Gamma_1 [O(1 - P(r = \varepsilon))O^{-1}] \tag{3.22}$$

$$= (O^{-1})^\dagger (1 - P(r = 0))^\dagger O^\dagger \Gamma_1 \; O(1 - P(r = 0))O^{-1} \tag{3.23}$$

$$= 0 \tag{3.24}$$

since $O^{\dagger}\Gamma_1 O = \Gamma_1$, which in turn comes from $M\Gamma_1 = -\Gamma_1 M$. Thus, we conclude that $P(r = r_0)$ describes unitary boundary conditions, so that

$$\psi_{D_1,-1/2} = U\ \psi_{D_2,1/2} \tag{3.25}$$

where U,which is an orthogonal matrix in our conventions, depends on the radial functions inside the monopole core, and consequently contains information about the core.

## IV. EFFECTIVE BOUNDARY CONDITIONS

In the previous section, we obtained the boundary conditions in terms of all the fields that exist in a given partial wave. But kinematically, only some of the fields can enter the monopole core and have a non-zero amplitude outside. We would like to get the boundary conditions in terms of these physically relevant fields. To find out which are the physically relevant fields, we have to look at the equations of motion outside the core. From Section II, we know that, for $E \approx 0$,

$$\partial_r \begin{pmatrix} \psi_{D_1,-1/2} \\ \psi_{D_2,1/2} \end{pmatrix} = \begin{pmatrix} 0 & A_+ \\ A_+^\dagger & 0 \end{pmatrix} \begin{pmatrix} \psi_{D_1,-1/2} \\ \psi_{D_2,1/2} \end{pmatrix} \tag{4.1}$$

where

$$A_+ = \left(\sqrt{J(J+1) - (\hat{r}\cdot\vec{T})^2 + 1/4}\ \mathbb{1} - \hat{T}^-\right)\hat{S}^+ \tag{4.2}$$

$$= \left(\sqrt{M(M+1) - \hat{Q}^2 + 1/4}\ \mathbb{1}\right)\hat{S}^+ \tag{4.3}$$

$A_+$ may be diagonalised just as $A_0$ was, and in the new basis, $M^2$ is diagonal instead of $\hat{r}\cdot\vec{T}$ --i.e. we have

$$\begin{pmatrix} \psi_{M_1^2,-1/2} \\ \psi_{M_2^2,1/2} \end{pmatrix} = \begin{pmatrix} S_+ & \psi_{D_1,-1/2} \\ S_+U_+ & \psi_{D_2,1/2} \end{pmatrix} \tag{4.4}$$

where $S_+$ and $U_+$ are defined analogously to $S_0$ and $U_0$ defined in Section II, and the equations of motion in this basis are

$$\partial_r \begin{pmatrix} \psi_{M_1^2,-1/2} \\ \psi_{M_2^2,1/2} \end{pmatrix} = \begin{pmatrix} 0 & D_+ \\ D_+ & 0 \end{pmatrix} \begin{pmatrix} \psi_{M_1^2,-1/2} \\ \psi_{M_2^2,1/2} \end{pmatrix} \tag{4.5}$$

This equation has to be interpreted carefully. Firstly, $\hat{Q}$ is diagonal in this basis as well, and it is clear that zeroes appear in $D_+$ whenever $q = \pm(m+1/2)$. Furthermore ,two states that are connected by non-zero eigenvalues of $D_+$ have the same $\hat{Q}$ and $M^2$ eigenvalues. Let us order the fields in equation 4.5 so that the fields with $(m+1/2)^2 = q^2$ are in the first 2p places.The remaining fields are arranged so that the entries $\sqrt{m(m+1) + 1/4 - q^2}$ in D+ connect the pair $\psi_{m,q,-1/2}$ to $\psi_{m,q,1/2}$ - i.e., we can write equation 4.5 as

$$\partial_r \begin{pmatrix} (\psi_{m_1^2}, q_1, -1/2)\, i \\ (\psi_{m_2^2}, q_2, 1/2)\, i \\ (\psi_{m^2}, q, -1/2)\, j \\ (\psi_{m^2}, q, 1/2)\, j \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_j \\ 0 & 0 & \alpha_j & 0 \end{pmatrix} \begin{pmatrix} (\psi_{m_1^2}, q_1, -1/2)\, i \\ (\psi_{m_2^2}, q_2, 1/2)\, i \\ (\psi_{m^2}, q, -1/2)\, j \\ (\psi_{m^2}, q, 1/2)\, j \end{pmatrix} \qquad 4.6$$

where i = 1, ---- ,p ,j = p+1, ---- ,n and n is the number of fields in the given partial wave. $\alpha_j$ is a diagonal positive definite matrix. The $\psi$ fields are explicitly labelled by their $\hat{Q}$ eigenvalues as well, just for clarity.The solutions to these equations are

$$A_i \equiv \left( \psi_{M_1^2,\, q_1,\, -1/2} \right)_i = a_i \qquad 4.7$$

$$B_i \equiv \left( \psi_{M_2^2,\, q_2,\, 1/2} \right)_i = b_i \qquad 4.8$$

$$C_j \equiv \left( \psi_{M^2,\, q,\, -1/2} + \psi_{M^2,\, q,\, 1/2} \right)_j$$

$$= c_j r^k \quad , \quad k > 0 \qquad 4.9$$

$$D_j \equiv \left( \psi_{M^2,\, q,\, -1/2} - \psi_{M^2,\, q,\, 1/2} \right)_j$$

$$= d_j \left( \frac{r_0}{r} \right)^k \quad , \quad k > 0 \qquad 4.10$$

where $a_i$, $b_i$, $c_j$ and $d_j$ are constants. It is clear that the amplitude for the fields $C_j$ are proportional to $r_0$ at the boundary, and hence in the limit $r_0 \to 0$, they cannot enter the core. On the other hand, the fields $D_j$ have a large amplitude at the core, but in the limit $r_0 \to 0$, they have a vanishingly small amplitude outside. Hence, the only physically relevant fields are $A_i$ and $B_i$. Since the equations of motion outside the core do not couple them to $C_j$ and $D_j$, if we can write the boundary conditions in terms of the $A_i$ and $B_i$ fields, we may drop the $C_j$ and $D_j$ fields from the problem altogether.

At the end of Section III, we obtained unitary boundary conditions in the $\hat{r}\cdot\vec{T}$ diagonal basis. Hence, we have

$$\psi_{M_1^2,-1/2} = S_+' U U_+^\dagger S_+'^\dagger \psi_{M_2^2,1/2}, \quad r = r_0 \tag{4.11}$$

$$= V \psi_{M_2^2,1/2}, \quad r = r_0 \tag{4.12}$$

where $S_+'$ and $U_+'$ are defined from $S_+$ and $U_+$ by a suitable reshuffling of the rows so that the fields are now ordered as in equation 4.6 and V is a unitary matrix.In terms of the A,B,C and D fields,4.12 may be written as

$$\begin{pmatrix} A \\ Cr_0 + D \end{pmatrix} = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix} \begin{pmatrix} B \\ Cr_0 - D \end{pmatrix} \tag{4.13}$$

Ignoring terms of $O(r_0)$, we get the following condition in terms of the physical fields

$$A = \{V_1 - V_2 (1 + V_4)^{-1} V_3\} B \tag{4.14}$$

which can be explicitly shown to be unitary. Hence, we may write

$$\psi_{M_1^2,-1/2} = U_{phys} \psi_{M_2^2,1/2}, \quad i = 1 \ . \ . \ . \ p \tag{4.15}$$

in terms of only the physically relevant fields. Our problem has been effectively reduced to physical fields interacting with the core via the unitary boundary condition 4.15.

We have yet a couple of technical points to mention. One is that we have been using $J^2$, $J_3$ eigen-states to describe the monopole-fermion interactions. This is a convenient basis inside the monopole core, because $\vec{J}$ is strictly conserved. But, in the spherically symmetric gauge that we are employing here, the physical angular momentum of the particles is given by $\vec{M}_+ = \vec{J} - \vec{I}(\hat{r})$. Hence, the physical particles are eigenstates of $|M_+^2, M_{+3}, I_3(\hat{r}), \hat{r}\cdot S, \hat{Q}>$ ($\equiv \psi_{phys, M_3, \hat{r}\cdot\vec{S}}$ ) which are related to the $|J^2, J_3, M^2, \hat{r}\cdot\vec{S}, \hat{Q}>$ eigenstates that we have used in 4.15, by the appropriate Clebsch-Gordan co-efficients--i.e. by a unitary transformation. Since both bases are diagonal in $M_+^2$ and Q, the physically relevant states in one basis are transformed to physically relevant states in the other basis. With our choice of basis ,the Clebsch-Gordon coefficients are real, so that $\psi_{phys, M_3, -1/2}$ = a linear combination of $|J^2, J_3, \hat{Q}, M^2, \hat{r}\cdot\vec{S} = -1/2\rangle$ states and $\psi_{phys, M_3, +1/2}$ = the same linear combination of $|J^2, J_3, \hat{Q}, M^2, \hat{r}\cdot\vec{S} = 1/2\rangle$ states. Furthermore,the relative phases between the different $\vec{J}$ eigenstates is +1 (since $\vec{J}$ has standard angular momentum phases. Hence, there is no ambiguity in writing the boundary conditions in the physical basis as

$$\psi_{phys,-1/2} = U'_{phys}\, \psi_{phys,1/2} \tag{4.16}$$

where $U'_{phys}$ is an orthogonal matrix. It should be noted however, that it was crucial to this transformation that the Clebsch-Gordon coefficients could be chosen to be real and positive,and that the relative phases between the different $\vec{J}$ eigenstates could be +1 simultaneously.

The second point to note is that if $U'_{phys}$ is not diagonal, an incoming physical particle scatters to a linear combination of outgoing physical particles. But, in most cases, $U'_{phys}$ turns out to be 1 dimensional and automatically diagonal.

## V. DISCUSSIONS AND CONCLUSIONS

In this section, we discuss how our results differ from the results using the step function approximation.

When the boundary conditions, in terms of the physically relevant fields are one-dimensional, we have

$$\psi_{in} = \pm \psi_{out} \tag{5.1}$$

in the step function approximation. After evolution through the radial function, the phase is still either +1 or -1, but any Green's function involving these fermions is only multiplied by $\pm 1$, which is unobservable when we square the amplitude to get the cross-section.

When the boundary conditions are not one-dimensional, we do not expect $U'_{phys}$ to be diagonal in general, unless it is due to some symmetry. Here, we have

$$\psi^i_{in} = U_{ij}(0)\, \psi^j_{out} \tag{5.2}$$

in the step-function approximation, and

$$\psi^{i}_{in} = U_{ij}\ (r_0)\ \psi^{j}_{out} \qquad (5.3)$$

after evolution through arbitrary radial functions. Here, a Green's function involving $\psi^{i}_{in}$ and $\psi^{j}_{out}$ is multiplied by $U_{ij}(0)$ in one case and $U_{ij}(r_0)$ in the other case, which may be numerically different. Hence, the cross-sections for $\psi^{i}_{in}$ scattering to $\psi^{j}_{out}$ are numerically different from the step-function approximation. Such processes, in principle, may be used to probe inside the monopole core, since $U_{ij}(r_0)$ is sensitive to the radial functions inside the core. However, we have not found such a case among the simplest examples that we have looked at.

Hence, in conclusion, we have analysed fermions in arbitrary representations of the gauge group interacting with arbitrary spherically symmetric monopoles, after including the radial functions inside the core. We find that in most simple cases, our results are exactly identical to the results using the step function approximation. But in certain cases, our results do differ and these cases can be used to probe the monopole core.

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